Interpolating real polynomials

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Interpolating sequences

Let $X$ be a set and $H$ a reproducing kernel Hilbert space of real functions defined on $X$, i.e. for all $x \in X$, there is a $K_x \in H$ such that

$$f(x) = \langle f, K_x \rangle.$$ 

We normalize the reproducing kernel and denote

$$\kappa_x = K_x / \| K_x \|.$$
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**Definition**

A sequence $\Lambda \subset X$ is an interpolating sequence for $H$ whenever

$$\sum_{\lambda \in \Lambda} |c_\lambda|^2 \simeq \| \sum_{\lambda \in \Lambda} c_\lambda \kappa_\lambda \|^2.$$
Let $\Lambda \subset \mathbb{R}$, then

**Definition**

A sequence of functions $\{f_\lambda(z) = \frac{\sin \pi(z-\lambda)}{\pi(z-\lambda)}\}_{\lambda \in \Lambda}$ is a Riesz sequence for the Paley Wiener space whenever,

$$\sum_{\lambda \in \Lambda} |c_\lambda|^2 \lesssim \left| \sum_{\lambda \in \Lambda} c_\lambda f_\lambda \right|^2 \lesssim \sum_{\lambda \in \Lambda} |c_\lambda|^2.$$

This implies that $\Lambda$ is uniformly separated.
The density of a interpolating sequences

There is a density that almost describes interpolating sequences

**Definition**

The upper Beurling-Landau density of a sequence $\Lambda \subset \mathbb{R}$ is

$$D^+(\Lambda) = \lim_{r \to \infty} \sup_{x \in \mathbb{R}} \frac{\#\{\Lambda \cap (x - r, x + r)\}}{2r}.$$
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\]

**Theorem (Beurling)**

A separated sequence \( \Lambda \subset \mathbb{R} \) is interpolating for PW if \( D^+(\Lambda) < 1 \). Moreover if \( \Lambda \) is interpolating then \( D^+(\Lambda) \leq 1 \).
Our setting

Let $\Omega$ be a smooth bounded strictly convex domain in $\mathbb{R}^d$. 

Let $P_n$ be the real polynomials of degree $n$. Let $dV$ be the normalized Lebesgue measure restricted to $\Omega$. We denote by $N_n = \dim(P_n)$. We endow $P_n$ with the norm given by $L_2^2(V)$. 

$\|p\|_2 = \int_{\Omega} |f(x)|^2 dV(x)$. 
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$$\|p\|^2 = \int_{\Omega} |f(x)|^2 \, dV(x).$$
Interpolating sequences

Let \( \Lambda = \{ \Lambda_n \}_n \subset \Omega \) be a sequence of finite sets of points of \( \Omega \subset \mathbb{R}^d \).

**Definition**

We say that \( \Lambda \) is an interpolating sequence if there is a constant \( C > 0 \) such that

\[
C^{-1} \sum_{\lambda \in \Lambda_n} |c_\lambda|^2 \leq \left| \sum_{\lambda \in \Lambda} c_\lambda \kappa^n_\lambda \right|^2 \leq C \sum_{\lambda \in \Lambda_n} |c_\lambda|^2,
\]

where \( \kappa^n_\lambda \) is the normalized reproducing kernel.

We are interested in the geometric distribution of points in \( \Lambda \).
Λ is an interpolating is equivalent to the two following properties.

\[
\sum_{\lambda \in \Lambda_n} \frac{|p(\lambda)|^2}{K_n(\lambda, \lambda)} \leq C \|p\|^2, \quad \forall p \in \mathcal{P}_n
\]

and for any sequence of sets of values \( \{v_\lambda\}_{\lambda \in \Lambda_v} \) there are polynomials \( p_n \in \mathcal{P}_n \) such that \( p_n(\lambda) = v_\lambda \) with

\[
\|p_n\|^2 \leq C \sum_{\lambda \in \Lambda_n} \frac{|v_\lambda|^2}{K_n(\lambda, \lambda)}.
\]
The “natural” normalization

The natural normalization is

\[ c_{\lambda,n} = \sup_{p \in P_n, \|p\|=1} |p(\lambda)|^2. \]
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This can be computed as follows. Take \( p_1, \ldots, p_{N_n} \) an orthonormal basis of \( \mathcal{P}_n \) and construct:

\[ K_n(z, w) = \sum_j p_j(z)p_j(w), \]

Moreover \( K_n \) is the reproducing kernel:

\[ p(z) = \int_\Omega K_n(z, w)p(w)\,dV(w), \quad \forall p \in \mathcal{P}_n \]
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\[ K_n(z, w) = \sum_j p_j(z)p_j(w), \]

\[ c_{\lambda,n} = K_n(\lambda, \lambda) \simeq \min \left( \frac{n^d}{\sqrt{d(\lambda)}}, n^{d+1} \right). \]
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Moreover $K_n$ is the reproducing kernel:

$$p(z) = \int_{\Omega} K_n(z, w)p(w) \, dV(w), \quad \forall p \in \mathcal{P}_n$$
The Plancherel-Polya sequences are a particular case of Carleson measures.

**Definition**

A sequence of measures in $\Omega$, $\mu_k$ is Carleson if there is a constant $C > 0$ such that

$$\int_{\Omega} |p|^2 \, d\mu_k \leq C\|p\|^2, \quad \forall p \in \mathcal{P}_k.$$ 

We have a geometric characterization of Carleson measures.
An anisotropic metric

In the ball there is an anisotopic distance given by

\[ d(x, y) = \arccos \left\{ \langle x, y \rangle + \sqrt{1 - |x|^2} + \sqrt{1 - |y|^2} \right\}. \]

This is the geodesic distance of the points in the sphere \( S^d \) defined as \( x' = (x, \sqrt{1 - |x|^2}) \) and \( y' = (x, \sqrt{1 - |x|^2}). \)
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If we consider balls \( B(x, r) \) in this distance they are comparable to a box (a product of intervals) which is of size \( R \) in the tangent directions and \( R^2 + R\sqrt{1 - |x|^2} \) in the normal direction.
The geometric characterization of the Carleson measures is the following:

**Theorem**

Let $\Omega$ be a ball. A sequence of measures $\mu_n$ is Carleson if there is a constant $C$ such that for all points $z \in \Omega$

$$\mu_n(B(z, 1/n)) \leq CV(B(z, 1/n)).$$
Bochner-Riesz type kernels

Proof.

The main ingredient in the proof is the existence of well localized kernels (the needlets of Petrushev and Xu), i.e. kernels $L_n(x, y)$ such that for an arbitrary $k$ there is a constant $C_k$ such that:

$$|L_n(x, y)| \leq C_k \frac{\sqrt{K_n(x, x)K_n(y, y)}}{(1 + nd(x, y))^k},$$

and moreover $L_n(x, x) \simeq K_n(x, x)$ and $L_n \in P_{2n}$ and reproduce the polynomials of degree $n$. 

We try to identify which is the critical density. We will use the following result:

**Theorem (Berman, Boucksom, Witt-Nyström)**

If $\mu$ is a Bernstein-Markov measure then

$$\frac{K_n(x, x)d\mu(x)}{N_n} \star \mu^{eq}.$$

The Bernstein-Markov condition is technical and it is satisfied when $\mu = \chi_{\Omega}dV$. The measure $\mu^{eq}$ is the equilibrium measure.
The equilibrium potential

**Definition**

Given a compact $K = \overline{\Omega} \subset \mathbb{R}^d$ and any $z \in \mathbb{C}^d$ one defines the Siciak-Zaharjuta equilibrium potential as

$$u_K(z) = \sup \left\{ \frac{\log |p(z)|}{\deg(p)} : \sup_K |p| \leq 1 \right\}.$$
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Then the equilibrium measure is defined as the Monge-Ampere of $u_K$

$$\mu^{eq} = (i\partial\bar{\partial}u_K)^d.$$

The equilibrium measure is a positive measure supported on $K$. 
What does $\mu^{eq}$ look like?

The measure $\mu^{eq}$ is a well-known object in pluripotential theory. In the examples we mentioned before it is well understood.

**Theorem (Bedford-Taylor)**

If $\Omega$ is an open bounded convex set in $\mathbb{R}^d$ then

$$d\mu^{eq}(x) \sim d_{euc}(x, \partial \Omega)^{-1/2} dV(x).$$
Main result

Theorem

If $\Lambda$ is an interpolating sequence for the polynomials in a bounded smooth strictly convex domain then

$$\limsup_{n \to \infty} \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \delta_\lambda \leq \mu_{eq}.$$

In particular, given any ball $B$ in $\Omega$ we have

$$\limsup_{n \to \infty} \frac{\#(\Lambda_n \cap B)}{N_n} \leq \mu_{eq}(B),$$

thus $\mu_{eq}$ is the Nyquist density.
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The Kantorovich-Wasserstein distance

Given a compact metric space $K$ we defines the K-W distance between two measures $\mu$ and $\nu$ supported in $K$ as

$$KW(\mu, \nu) = \inf_{\rho} \iint_{K \times K} d(x, y) \, d\rho(x, y),$$

where $\rho$ is an admissible measure, i.e. the marginals of $\rho$ are $\mu$ and $\nu$ respectively.
The Kantorovich-Wasserstein distance

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where $\rho$ is an admissible measure, i.e. the marginals of $\rho$ are $\mu$ and $\nu$ respectively. Alternatively:

$$KW(\mu, \nu) = \inf_{\rho} \int\int_{K \times K} d(x, y) d|\rho|(x, y),$$

where $\rho$ is an admissible complex measure, i.e. the marginals of $\rho$ are $\mu$ and $\nu$ respectively.
The complex transport plan

The K-W distance metrizes the weak-* convergence. We want to prove that

$$KW(b_n, \sigma_n) \to 0,$$

where $b_n \leq K_n(x, x) dV(x)/N_n$ is smaller than the Bergman measure and

$$\sigma_n = \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \delta_\lambda$$
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\[ \sigma_n = \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \delta_\lambda \]

The transport plan \( \rho_n \) that is convenient to estimate is:

\[ \rho_n(x, y) = \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \delta_\lambda(y) \times g_\lambda(x) \frac{K_n(\lambda, x)}{\sqrt{K_n(\lambda, \lambda)}} \, dV(x), \]

where \( g_\lambda \) is the biorthogonal basis to \( \left\{ \frac{K_n(\lambda, x)}{\sqrt{K_n(\lambda, \lambda)}} \right\}_{\lambda \in \Lambda_n} \) in the space \( \mathcal{F}_n \subset \mathcal{P}_n \) spanned by \( \{ \kappa_\lambda, \lambda \in \Lambda_n \} \).
The complex transport plan

The two marginals of $\rho_n$ are

$$\nu_n := \frac{1}{N_n} K_n(x, x) \, dV(x) \leq \frac{1}{N_n} K_n(x, x) \, dV(x) \overset{*}{\to} \mu^{eq}$$
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and

$$KW(\nu_n, \sigma_n) \leq \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \int_\Omega d(\lambda, x) |g_\lambda(x)| \frac{|K_n(\lambda, x)|}{\sqrt{K_n(\lambda, \lambda)}} \, dV(x).$$
The complex transport plan

The two marginals of $\rho_n$ are

- $\nu_n := \frac{1}{N_n} \mathcal{K}_n(x, x) \, dV(x) \leq \frac{1}{N_n} K_n(x, x) \, dV(x) \xrightarrow{\ast} \mu^{eq}$
- $\sigma_n := \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \delta_{\lambda}$

and

$$KW(\nu_n, \sigma_n) \leq \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \int_{\Omega} d(\lambda, x) |g_{\lambda}(x)| \frac{|K_n(\lambda, x)|}{\sqrt{K_n(\lambda, \lambda)}} \, dV(x).$$

Thus

$$KW^2(\nu_n, \sigma_n) \lesssim \frac{1}{N_n} \iint d^2(x, y) |K_n(x, y)|^2 \, dV(x) \, dV(y).$$
An off-diagonal estimate

Given a bounded function $f$ on $M$ we denote by $T_f$ be the Toeplitz operator on $\mathcal{P}_n \cap L^2(\Omega)$ with symbol $f$, i.e. $T_f := \Pi_n \circ f$. where $\Pi_n$ denotes the orthogonal projection from $L^2(\Omega)$ to $\mathcal{P}_n$. 

It can be easily computed:

$$\text{Tr} T_f^2 - \text{Tr} T_f^2 = \frac{1}{2} \int_{\Omega} \int_{\Omega} \left( f(x) - f(y) \right)^2 |K_n(x,y)|^2 dV(x) dV(y).$$

Now, setting $f := x^i$ we observe than on $\mathcal{P}_n - 1$, $T_f(p) = x^i p$.

Therefore:

$$\text{Tr} T_f^2 - \text{Tr} T_f^2 = 0 \text{ on } \mathcal{P}_n - 2.$$

Therefore:

$$\text{Tr} T_f^2 - \text{Tr} T_f^2 = O(k_n - 1)$$

and

$$K_W^2(\nu_n, \sigma_n) \lesssim 1/n.$$
An off-diagonal estimate

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$$\text{Tr } T_f^2 - \text{Tr } T_f = \frac{1}{2} \int_{\Omega \times \Omega} (f(x) - f(y))^2 |K_n(x, y)|^2 dV(x)dV(y).$$
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$$\text{Tr } T_f^2 - \text{Tr } T_{f^2} = \frac{1}{2} \int_{\Omega \times \Omega} (f(x) - f(y))^2 |K_n(x, y)|^2 dV(x)dV(y).$$

Now, setting $f := x_i$ we observe than on $\mathcal{P}_{n-1}$, $T_f(p) = x_ip$. Therefore $T_{f^2} - T_f^2 = 0$ on $\mathcal{P}_{n-2}$. Therefore:

$$\text{Tr } T_f^2 - \text{Tr } T_{f^2} = O(k^{n-1})$$

and

$$KW^2(\nu_n, \sigma_n) \lesssim \frac{1}{n}.$$
Some extensions

There are many extensions of this result. Of special interest: Let $M$ be a compact smooth algebraic variety in $\mathbb{R}^m$. We endow the space of polynomials $P_n$ restricted to $M$ with the $L^2$ norm with respect to the Lebesgue measure. We define interpolating sequences $\Lambda$ as before.
There are many extensions of this result. Of special interest:
Let $M$ be a compact smooth algebraic variety in $\mathbb{R}^m$.
We endow the space of polynomials $\mathcal{P}_n$ restricted to $M$ with the $L^2$ norm with respect to the Lebesgue measure. We define interpolating sequences $\Lambda$ as before.

Theorem

If $\Lambda$ is an interpolating sequence for the polynomials then

$$\limsup_{n \to \infty} \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \delta_\lambda \leq \mu^{eq}.$$  

The equilibrium measure in this setting is comparable to the Lebesgue measure.